A REFLEXIVE BANACH SPACE WHOSE ALGEBRA OF OPERATORS IS NOT A GROTHENDIECK SPACE

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Abstract. By a result of Johnson, the Banach space $F = \big(\bigoplus_{n=1}^{\infty} \ell_1^n\big)_{\ell_\infty}$ contains a complemented copy of ℓ_1. We identify F with a complemented subspace of the space of (bounded, linear) operators on the reflexive space $\big(\bigoplus_{n=1}^{\infty} \ell_1^n\big)_{\ell_p}$ ($p \in (1,\infty)$), thus solving negatively the problem posed in the monograph of Diestel and Uhl which asks whether the space of operators on a reflexive Banach space is Grothendieck.

1. Introduction

A Banach space E is *Grothendieck* if weak* convergent sequences in E^* converge weakly. Certainly, every reflexive Banach space is Grothendieck. Notable examples of non-reflexive Grothendieck spaces are $C(K)$-spaces for extermally disconnected compact spaces K ([4]) and the Hardy space H^∞ of bounded holomorphic functions on the unit disc ([1]). Diestel and Uhl wrote in their famous monograph [3, p. 180]:

> *Finally, there is some evidence (Akemann [1967], [1968]) that the space $\mathscr{L}(H;H)$ of bounded linear operators on a Hilbert space is a Grothendieck space and that more generally the space $\mathscr{L}(X;X)$ is a Grothendieck space for any reflexive Banach space X.*

The question whether the space of (bounded, linear) operators on a reflexive Banach space is Grothendieck was raised also by Soybaş ([7]). Pfitzner proved in [6] that C*-algebras have the so-called *Pełczyński's property* (V) which for dual Banach spaces is equivalent to being a Grothendieck space (cf. [2, Exercise 12, p. 116]). In particular, von Neumann algebras are Grothendieck spaces which confirms that the space of operators on a Hilbert space is Grothendieck. It is known that duals of spaces with property (V) are weakly sequentially complete. We shall present an example of a reflexive Banach space E so that $\mathscr{B}(E)$ fails to be Grothendieck, giving thus a negative answer to the above-mentioned problem. To do this, we require a result of Johnson which asserts that the Banach space $F = \big(\bigoplus_{n=1}^{\infty} \ell_1^n\big)_{\ell_\infty}$ contains a complemented copy of ℓ_1 (cf. Remark after Theorem 1 in [5]), so it is not a Grothendieck space.

By an *operator* we understand a bounded, linear operator acting between Banach spaces. The space $\mathscr{B}(E_1, E_2)$ of operators acting between spaces E_1 and E_2 is a Banach space when endowed with the operator norm. We write $\mathscr{B}(E)$ for $\mathscr{B}(E,E)$. Let $p \in [1,\infty]$. We denote by $(\bigoplus_{n=1}^{\infty} E_n)_{\ell_p}$ the ℓ_p-sum of a sequence $(E_n)_{n=1}^{\infty}$ of Banach spaces. We identify elements of $\mathscr{B}((\bigoplus_{n=1}^{\infty} E_n)_{\ell_p})$ with *matrices* $(T_{ij})_{i,j\in\mathbb{N}}$, where $T_{ij} \in \mathscr{B}(E_j, E_i)$ ($i,j \in \mathbb{N}$). Let $(e_n)_{n=1}^{\infty}$ be the canonical basis of ℓ_1. For each $n \in \mathbb{N}$ we denote $\ell_1^n = \operatorname{span}\{e_1, \dots, e_n\}$.

2. The result

Main result. *Let $p \in (1,\infty)$ and consider the reflexive Banach space $E = \big(\bigoplus_{n=1}^\infty \ell_1^n\big)_{\ell_p}$. Then $\mathscr{B}(E)$ is not a Grothendieck space.*

Proof. Recall that $F = \big(\bigoplus_{n=1}^\infty \ell_1^n\big)_{\ell_\infty}$ contains a complemented copy of ℓ_1. To complete the proof it is enough to embed F as a complemented subspace of $\mathscr{B}(E)$.

One may identify ℓ_1^n with a 1-complemented subspace of $\mathscr{B}(\ell_1^n)$ via the mapping

$$e_k \mapsto e_k \otimes e_1^* \ \ (k \leqslant n,\, n \in \mathbb{N}),$$

where e_1^* stands for the coordinate functional associated to e_1. Consequently, the space $D = (\bigoplus_{n=1}^\infty \mathscr{B}(\ell_1^n))_{\ell_\infty}$ contains a complemented subspace isomorphic to F. Let $\Delta\colon D \to \mathscr{B}(E)$ be the *diagonal embedding*, that is, $\Delta((T_n)_{n=1}^\infty) = \operatorname{diag}(T_1, T_2, \ldots)$ $((T_n)_{n=1}^\infty \in D)$; this map is well-defined since the decomposition of E into the subspaces $\ell_1^1, \ell_1^2, \ldots$ is unconditional.

It is enough to notice that Δ has a left-inverse $\Xi\colon \mathscr{B}(E) \to D$ given by

$$\Xi(T_{ij})_{i,j\in\mathbb{N}} = (T_{ii})_{i=1}^\infty \ \ ((T_{ij})_{i,j\in\mathbb{N}} \in \mathscr{B}(E)),$$

which is bounded. To this end, we shall perform a construction inspired by a trick of Tong (*cf.* [8, Theorem 2.3] and its proof). With each operator $T = (T_{ij})_{i,j\in\mathbb{N}} \in \mathscr{B}(E)$ we shall associate a sequence $\big(S^{(n)}\big)_{n=1}^\infty$ of finite-rank perturbations of T such that for each $n \in \mathbb{N}$ we have $\|S^{(n)}\| \leqslant \|T\|$ and the matrix of $S^{(n)}$ agrees with the matrix of the diagonal operator $\operatorname{diag}(-T_{11}, \ldots, -T_{nn}, 0, 0, \ldots)$ at entries (i,j) with $i \leqslant n$ or $j \leqslant n$. This will immediately yield that

$$\|\Xi(T)\| = \sup_{n\in\mathbb{N}} \|T_{nn}\| = \sup_{n\in\mathbb{N}} \|-S_{nn}^{(n)}\| \leqslant \sup_{n\in\mathbb{N}} \|S^{(n)}\| \leqslant \|T\|.$$

Define operators $T_{\mathsf{k}}, T_{\mathsf{r}}$ which have the same columns and rows as T respectively, except the first ones, where we instead set $(T_{\mathsf{k}})_{i1} = -T_{i1}$ and $(T_{\mathsf{r}})_{1j} = -T_{1j}$ for $i,j \in \mathbb{N}$ (these are indeed elements of $\mathscr{B}(E)$ as rank-one perturbations of T). Certainly, $\|T\| = \|T_{\mathsf{k}}\| = \|T_{\mathsf{r}}\|$ and the norm of $S = (T_{\mathsf{k}} + T_{\mathsf{r}})/2$ does not exceed the norm of T. Arguing similarly, we observe that $\|(S_{\mathsf{k}}^{(n)} + S_{\mathsf{r}}^{(n)})/2\| \leqslant \|T\|$, where $S^{(1)} = S$ and $S^{(n+1)} = (S_{\mathsf{k}}^{(n)} + S_{\mathsf{r}}^{(n)})/2$ $(n \in \mathbb{N})$. Consequently, $\big(S^{(n)}\big)_{n=1}^\infty$ is the desired sequence. □

Remark. The space $\mathscr{B}(E)$ shares with the space of operators on a Hilbert space a number of common properties. For instance, since E has a Schauder basis, $\mathscr{B}(E)$ can be identified with the bidual of $\mathscr{K}(E)$, the space of compact operators on E. Nonetheless, E is plainly not superreflexive and $\mathscr{B}(E)$ fails to have weakly sequentially dual for the obvious reason ℓ_∞ embeds into $\mathscr{B}(E)^*$. We conjecture that the space of operators on a superreflexive space is Grothendieck (or at least it has weakly sequentially complete dual).

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